MODULARITY LIFTING BEYOND THE TAYLOR–WILES METHOD. II

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Contents

1. Introduction

In a previous paper [CG], we showed how one could generalize Taylor–Wiles modularity lifting theorems [Wil95, TW95] to contexts beyond those in which the automorphic forms in question arose from the middle degree cohomology of Shimura varieties; in particular, to contexts in which the relevant automorphic forms contributed to cohomology in exactly two degrees. In this sequel, we extend our method to the general case in which Galois representations are expected to occur in cohomology, *contingent* on the (as yet unproven) existence of certain Galois representations. Our method currently applies (in principle) in two contexts:

Betti To Galois representations conjecturally arising from tempered π of cohomological type associated to G, where G is reductive with a maximal compact K, maximal $\mathbf{Q}$-split torus A, and $l_0 = \mathrm{rank}(G) - \mathrm{rank}(K) - \mathrm{rank}(A)$ is arbitrary,

Coherent To Galois representations conjecturally arising from tempered π associated to G, where (G, X) is a Shimura variety over a totally real field F, and such that π_v at all infinite places is either a discrete series or a holomorphic limit of discrete series.

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We do not address the problem of constructing Galois representations. Note that our methods require Galois representations associated to torsion classes which do not necessarily lift to characteristic zero, so the forthcoming results of [HLTT] are not sufficient for our needs, even if we restrict our attention to CM fields.

The following results are a sample of what can be shown by these methods in case **Betti**, assuming (Conjecture A of § 6.1) the existence of Galois representations in appropriate degrees satisfying the expected properties.

Theorem 1.1. *Assume Conjecture A. Let F be any number field, and let E be an elliptic curve over F. Then the following hold:*

(1) E is potentially modular.
(2) The Sato–Tate conjecture is true for E.

In the context of the case **Coherent**, it transpires that our results are *still* conditional on conjectures concerning the existence of Galois representations, since the usual methods for constructing representations (using congruences) only work with Hecke actions on $H^0(X,\mathscr{E})$ rather than $H^i(X,\mathscr{E})$ for $i>0$. In this paper, we address the commutative algebra issues necessary to handle the coherent case, but we omit any specific conjectural applications, instead mentioning that we intend to come back to the coherent case in future work (where, in a limited number of cases, we have some unconditional results).

The proof of Theorem 1.1 relies on the following ingredients. The first ingredient consists of the usual techniques in modularity lifting (the Taylor–Wiles–Kisin method) as augmented by Taylor's Ihara's Lemma avoidance trick [Tay08]. The second ingredient is to observe that these arguments *continue to hold* in a more general situation, *provided* that one can show that there is "enough" cohomology. This is the philosophy explained in [CG] and amounts to giving a lower bound on the depth of certain patched Hecke modules. Finally, one can obtain such a lower bound by a commutative algebra argument, *assuming* that the relevant cohomology occurs only in a certain range of length l_0. Conjecture A amounts to assuming both the existence of Galois representations together with the vanishing of cohomology (localized at an appropriate $\mathfrak{m}$) outside a given range.

We now state the (conditional) modularity lifting theorem used to prove the results above. Let $\mathcal{O}$ denote the ring of integers in a finite extension of $\mathbf{Q}_p$, let ϖ be a uniformizer of $\mathcal{O}$, and let $\mathcal{O}/\varpi=k$ be the residue field. Recall that a representation $\mathrm{Gal}(\mathbf{C}/\mathbf{R})\to\mathrm{GL}_n(\mathcal{O})$ is *odd* if the image of c has trace in $\{-1,0,1\}$.)

Theorem 1.2. *Assume Conjecture A. Let $F/\mathbf{Q}$ be an arbitrary number field, and n a positive integer. Let $p>n$ be unramified in F. Let*

$$r: G_F\to\mathrm{GL}_n(\mathcal{O})$$

be a continuous Galois representation unramified outside a finite set of primes. Denote the mod-ϖ reduction of r by $\overline{r}: G_F\to\mathrm{GL}_n(k)$. Suppose that

(1) If $v|p$, the representation $r|D_v$ is crystalline.
(2) $\dim_{\mathbf{Q}_p}\mathrm{gr}^i(r\otimes_{\mathbf{Q}_p}B_{\mathrm{DR}})^{\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)}=0$ unless $i\in\{0,1,\ldots,n-1\}$, in which case it has dimension 1.
(3) The restriction of $\overline{r}$ to $F(\zeta_p)$ is absolutely irreducible, and the field $F(\mathrm{ad}^0(\overline{r}))$ does not contain $F(\zeta_p)$.
(4) In the terminology of [CHT08], Definition 2.5.1, $\overline{r}$ is big.

(5) If $v|\infty$ is any real place of F, then $r|G_{F_v}$ is odd.
(6) If r is ramified at a prime x, then $r|I_x$ is unipotent. Moreover, if, futhermore, $\overline{r}$ is unramified at x, then $N(x) \equiv 1 \mod p$.
(7) Either:
(a) There exists a cuspidal automorphic representation π_0 of $\mathrm{GL}_n(\mathbb{A}_F)$ such that: $\pi_{0,v}$ has trivial infinitesimal character for all $v|\infty$, good reduction at all $v|p$, and the p-adic Galois representation $r_p(\pi)$ both satisfies condition 6 and the identity $\overline{r}_p(\pi) = \overline{r}$.
(b) $\overline{r}$ is Serre modular of minimal level $N(\overline{r})$, and r is ramified only at primes which ramify in $\overline{r}$.

Then r is modular, that is, there exists a regular algebraic cusp form π for $\mathrm{GL}_n(\mathbb{A}_F)$ with trivial infinitesimal character such that $L(r,s) = L(\pi,s)$.

As in [CG], condition 7b is only a statement about the existence of a mod-p cohomology class of level $N(\overline{r})$, not the existence of a characteristic zero lift; this condition is the natural generalization of Serre's conjecture. On the other hand, the usual strategy for proving potential modularity usually proceeds by producing characteristic zero lifts which are not minimal, and thus condition 7a will be useful for applications. If conditions 1, 2, and 3 are satisfied, then conditions 5 and 6 are satisfied after a solvable extension which is unramified at p. Moreover, *if* $\overline{r}$ admits an automorphic lift with trivial infinitesimal character and good reduction at p, then condition 7a is also satisfied after a solvable extension which is unramified at p. Condition 7b, however, is not obviously preserved under cyclic base change.

Note that it will be obvious to the expert that our methods will allow for (conditional) generalizations of these theorems to other contexts (for example, varying the weight) but we have contented ourselves with the simplest possible statements necessary to deduce Theorem 1.1. For a similar reason, we assume that $\overline{r}$ is big rather than adequate (in the sense of [Tho11]).

We caution, however, that several techniques are *not* available in this case, in particular, the lifting techniques of Ramakrishna and Khare–Wintenberger require that $l_0 = 0$.

The main ideas of this paper have been worked out for some time, but the delay in releasing this preprint has been caused by our attempts to provide unconditional results in several **Coherent** cases (we have some partial results in this direction which will hopefully appear in future work).

2. Acknowledgements

Several of the ideas in this paper go back to the writing of [CG] and even before then; conversations with Matthew Emerton have certainly helped inspire some of the philosophy behind [CG] and this paper. Some of the ideas were also discovered when both of the authors were members of the Institute for Advanced Study in 2010-2011. In September 2012, we received a communication from David Hansen indicating that he had independently discovered several of the commutative algebra ideas behind this paper (in the **Betti** case), and noted their possible application to minimal modularity lifting theorems, following [CG]; in our reply, we informed him that we were in process of finishing this paper. He has subsequently posted a short sketch of his approach to the arXiv as the preprint [Han12], which aims to prove a minimal ordinary modularity lifting theorem for GL(2) over an arbitrary number field. It should be noted, however, that several of the assertions in that paper are inconsistent with

the fact that the appropriate local ordinary deformation rings at p (for arbitrary number fields F) are not, in general, formally smooth.

Notation

In this paper, $\mathcal{O}$ will denote the ring of integers in a finite extension K of $\mathbf{Q}_p$. We let ϖ denote a uniformizer in $\mathcal{O}$ and let $k = \mathcal{O}/\varpi$ be the residue field. We denote by $\mathcal{C}_{\mathcal{O}}$ the category of complete Noetherian local $\mathcal{O}$-algebras with residue field k. If G is a group and $\chi : G \to k^\times$ is a character, we denote by $\langle \chi \rangle : G \to \mathcal{O}^\times$ the Teichmüller lift of χ.

If F is a field we let G_F denote the Galois group $\mathrm{Gal}(\overline{F}/F)$ for some choice of algebraic closure $\overline{F}/F$. We let $\epsilon : G_F \to \mathbf{Z}_p^\times$ denote the p-adic cyclotomic character. If F is a number field and v is a prime of F, we let $\mathcal{O}_v$ denote the ring of integers in the completion of F at v and we let π_v denote a uniformizer in $\mathcal{O}_v$. We denote G_{F_v} by G_v and let $I_v \subset G_v$ be the inertia group. We also let $\mathrm{Frob}_v \in G_v/I_v$ denote the *arithmetic* Frobenius. We let $\mathrm{Art} : F_v^\times \to W_{F_v}^{\mathrm{ab}}$ denote the local Artin map, normalized to send uniformizers to geometric Frobenius lifts. If R is a ring and $\alpha \in R^\times$, we let $\lambda(\alpha) : G_v \to R^\times$ denoted the unramified character which sends Frob_v to α, when such a character exists. We let $\mathbb{A}_F$ and $\mathbb{A}_F^\infty$ denote the adeles and finite adeles of F respectively. If $F = \mathbf{Q}$, we simply write $\mathbb{A}$ and $\mathbb{A}^\infty$.

If P is a bounded complex of S-modules for some ring S, then we let $H^*(P) = \oplus_i H^i(P)$. Any map $H^*(P) \to H^*(P)$ will be assumed to be degree preserving.

If R is a local ring, we will sometimes denote the maximal ideal of R by $\mathfrak{m}_R$.

3. Some Commutative Algebra

The general difficulty (as explained in [CG]) in proving that $R_\infty = \mathbf{T}_\infty$ is to show that there are *enough* modular Galois representations. If the cohomology we are interested in occurs in a range of degrees of length l_0, then we would like to show that in at least one of these degrees that the associated modules H_N (which are both Hecke modules and modules for the group rings $S_N := \mathcal{O}[(\mathbf{Z}/p^N\mathbf{Z})^q]$) compile, in a Taylor–Wiles patching process, to form a module of codimension l_0 over the completed group ring $S_\infty := \mathcal{O}[[(\mathbf{Z}_p)^q]]$. The problem then becomes to find a find a suitable notion of "codimension l_0" for modules over a local ring that

(1) is well behaved for non-reduced quotients of power series rings over $\mathcal{O}$ (like S_N),
(2) can be established for the spaces H_N in question,
(3) compiles well in a Taylor–Wiles system.

It turns out to be more effective to patch together a series of complexes D_N of length l_0 whose cohomology computes the cohomology of $\Gamma_1(Q_N)$ localized at $\mathfrak{m}$. The limit of these patched complexes will then turn out to be a length l_0 resolution of an associated patched module.

It will be useful to prove the following lemmas.

Lemma 3.1. *Let S be a Noetherian local ring. If N is an S-module with depth n, and $0 \neq M \subseteq N$, then* $\dim(M) \geq n$.

Proof. Let $\mathfrak{p}$ be an associated prime of M (and hence of N). Then $\mathfrak{p}$ is the annihilator of some $0 \neq m \in M$, and it suffices to prove the result for M replaced by $mS \subset M$. On the

other hand, for a Noetherian local ring, one has the inequality (see [Mat86], Theorem 17.2)

$$n = \operatorname{depth}(N) \leq \min_{\operatorname{Ass}(N)} \dim S/\mathfrak{p} \leq \dim(M).$$

$\square$

We deduce from this the following:

Lemma 3.2. *Let $l_0 \geq 0$ be an integer and let S be a complete Noetherian regular local ring of dimension $n \geq l_0$. Let P be a complex which is concentrated in degrees $0, \ldots, l_0$. Then $\operatorname{codim}(H^*(P)) \leq l_0$, and moreover, if equality occurs, then:*

(1) P is a resolution of $H^{l_0}(P)$,
(2) $H^{l_0}(P)$ has depth $n - l_0$ and has projective dimension l_0.

Proof. Let $m \leq l_0$ denote the smallest integer such that $H^m(P) \neq 0$. Consider the complex:

$$P^0 \to P^1 \to \ldots \to P^m.$$

By assumption, this complex is exact until the final term, and thus it is the resolution of a certain module $K^m := P^m/\operatorname{Im}(P^{m-1})$. It follows that the projective dimension of K^m is $\leq m$. On the other hand, we see that

$$H^m(P) = \ker(P^m)/\operatorname{Im}(P^{m-1}) \subseteq K^m,$$

and thus

$$\operatorname{codim}(H^m(P)) = n - \dim(H^m(P)) \leq n - \operatorname{depth}(K^m) = \operatorname{proj.dim}(K^m) \leq m,$$

where the central inequality is Lemma 3.1, and the second equality is the Auslander-Buchsbaum formula.

Suppose that $\operatorname{codim}(H^*(P)) \geq l_0$. Then it follows from the argument above that the smallest m for which $H^m(P)$ is non-zero is $m = l_0$, that $\operatorname{codim}(H^{l_0}(P)) = l_0$, that P is a resolution of $H^{l_0}(P)$, and that $\operatorname{proj.dim}(H^{l_0}(P)) = l_0$, completing the argument. $\square$

3.1. **Patching.** We establish in this section an abstract Taylor–Wiles style patching result which may be viewed as an analogue of Theorem 2.1 of [Dia97] and Proposition 2.3 of [CG], but also including refinements due to Kisin [Kis09].

Theorem 3.3. *Let q and j be non-negative integers with $q + j \geq l_0$, and let $S_\infty = \mathcal{O}[[(\mathbf{Z}_p)^q]]$. For each integer $N \geq 0$, let $S_N := \mathcal{O}[\Delta_N]$ with $\Delta_N := (\mathbf{Z}/p^N\mathbf{Z})^q$. For each $N \geq M \geq 0$ and each ideal I of $\mathcal{O}$, we regard S_N/I (and in particular, $\mathcal{O}/I = S_0/I$) as a quotient of S_M via the quotient map $\Delta_M \twoheadrightarrow \Delta_N$ and reduction modulo I.*

(1) Let R_∞ be an object of $\mathcal{C}_\mathcal{O}$ of Krull dimension $1 + j + q - l_0$.
(2) Let R be an object of $\mathcal{C}_\mathcal{O}$, let H be an R-module.
(3) Let T be a complex of finite dimensional k-vector spaces concentrated in degrees $0, \ldots, l_0$ together with a differential $d = 0$.

Let $\mathcal{O}^\square = \mathcal{O}[[z_1, \ldots, z_j]]$ and for each $\mathcal{O}$-module M, we let $M^\square := M \otimes_\mathcal{O} \mathcal{O}^\square$. We regard $\mathcal{O}$ as a $\mathcal{O}^\square$-algebra via the map sending each z_i to 0. Suppose that, for each integer $N \geq 1$, D_N is a perfect complex of S_N-modules with the following properties:

(a) There is an isomorphism $D_N \otimes_{S_N} S_N/\mathfrak{m}_{S_N} \simeq T$.

(b) For each $M \geq N \geq 0$ with $M \geq 1$ and each $n \geq 1$, the action of S_M on the cohomology of the complex $D_M^\square \otimes_{S_M} S_N/\varpi^n$ extends to an action of $R_\infty \otimes_{\mathcal{O}} S_M^\square$. If, in addition, $N \geq N' \geq 0$ and $n \geq n' \geq 1$, then the natural map $H^*(D_M^\square \otimes_{S_M} S_N/\varpi^n) \to H^*(D_M^\square \otimes_{S_M} S_{N'}/\varpi^{n'})$ is compatible with the R_∞-actions.

(c) For M, N and n as above, the image of $S_M^\square$ in $\mathrm{End}_{\mathcal{O}}(H^*(D_M^\square \otimes_{S_M} S_N/\varpi^n))$ is contained in the image of R_∞.

(d) For each $N \geq 1$, there is a surjective map $\phi_N : R_\infty \to R$, and for each $n \geq 1$ we are given an isomorphism

$$H^{l_0}(D_N^\square \otimes_{S_N^\square} \mathcal{O}/\varpi^n) = H^{l_0}(D_N^\square \otimes_{S_N} \mathcal{O}/\varpi^n) \otimes_{\mathcal{O}^\square} \mathcal{O} \simeq H/\varpi^n$$

of R_∞-modules where R_∞ acts on H/ϖ^n via ϕ_N. Moreover, these isomorphisms are compatible for fixed N and varying n.

Let $\mathfrak{a} \subset S_\infty^\square$ denote the kernel of the map $S_\infty^\square \to \mathcal{O}$ sending each element of $(\mathbf{Z}_p)^q$ to 1 and each z_i to 0. Then the following holds: there is a perfect complex $P_\infty^\square$ of finitely generated $S_\infty^\square$-modules concentrated in degrees $0, \ldots, l_0$ such that

(i) The complex $P_\infty^\square$ is a projective resolution, of minimal length, of its top degree cohomology $H^{l_0}(P_\infty^\square)$.

(ii) There is an action of $R_\infty \widehat{\otimes}_{\mathcal{O}} S_\infty^\square$ on $H^{l_0}(P_\infty^\square)$ extending the action of $S_\infty^\square$ and such that $H^{l_0}(P_\infty^\square)$ is a finite R_∞-module.

(iii) The R_∞-depth of $H^{l_0}(P_\infty)$ is equal to $1 + j + q - l_0 (= \dim R_\infty)$.

(iv) There is a surjection $\phi_\infty : R_\infty \twoheadrightarrow R$ and an isomorphism $\psi_\infty : H^{l_0}(P_\infty^\square)/\mathfrak{a} \xrightarrow{\sim} H$ of R_∞-modules where R_∞ acts on H via ϕ_∞.

Proof. For each $N \geq 1$, let $\mathfrak{a}_N$ denote the kernel of the natural surjection $S_\infty \twoheadrightarrow S_N$ and let $\mathfrak{b}_N$ denote the open ideal of $S_\infty^\square$ generated by ϖ^N, $\mathfrak{a}_N$ and $(z_1^N, \ldots, z_j^N)$. Choose a sequence of open ideals $(\mathfrak{d}_N)_{N \geq 1}$ of R such that

- $\mathfrak{d}_N \supset \mathfrak{d}_{N+1}$ for all $N \geq 1$;
- $\cap_{N \geq 1} \mathfrak{d}_N = (0)$;
- $\varpi^N R \subset \mathfrak{d}_N \subset \varpi^N R + \mathrm{Ann}_R(H)$ for all N.

Define a *patching datum of level* N to be a 3-tuple (ϕ, ψ, P) where

- $\phi : R_\infty \twoheadrightarrow R/\mathfrak{d}_N$ is a surjection in $\mathcal{C}_{\mathcal{O}}$;
- P is a perfect complex of $S_\infty/(\mathfrak{a}_N + \varpi^N)$-modules such that $P \otimes S_\infty/\mathfrak{m}_{S_\infty} \simeq T$;
- For each $N \geq N' \geq 0$, each $N \geq n' \geq 1$ and each ideal I of $\mathcal{O}^\square$ with $(z_1^N, \ldots, z_j^N) \subset I \subset (z_1, \ldots, z_j)$, the cohomology groups $H^i(P \otimes_{S_\infty} S_{N'} \otimes_{\mathcal{O}} \mathcal{O}^\square/(I + \varpi^{n'}))$ carry an action of $R_\infty \widehat{\otimes}_{\mathcal{O}} S_\infty^\square$ extending the action of $S_\infty^\square$, and these actions are compatible for varying N', n' and I;
- $\psi : H^{l_0}(P^\square \otimes_{S_\infty^\square} S_\infty^\square/(\mathfrak{a} + \varpi^N)) \xrightarrow{\sim} H/\varpi^N H$ is an isomorphism of R_∞ modules (where R_∞ acts on $H/\varpi^N H$ via ϕ). (Note that ψ then gives rise to an isomorphism of R_∞-modules between $H^{l_0}(P^\square \otimes_{S_\infty^\square} S_\infty^\square/(\mathfrak{a} + \varpi^{n'}))$ and $H/\varpi^{n'} H$ for each $N \geq n' \geq 1$.)

We say that two such 3-tuples (ϕ, ψ, P) and (ϕ', ψ', P') are isomorphic if $\phi = \phi'$ and there is an isomorphism of complexes $P \xrightarrow{\sim} P'$ of S_∞-modules inducing isomorphisms of $R_\infty \widehat{\otimes}_{\mathcal{O}} S_\infty$-modules on cohomology which are compatible with ψ and ψ' in degree l_0. We note that, up to isomorphism, there are finitely many patching data of level N. (This follows from the fact that R_∞ and S_∞ are topologically finitely generated, and that T is finite.) If D is a patching

datum of level N and $1 \le N' \le N$, then D gives rise to a patching datum of level N' in an obvious fashion. We denote this datum by $D \mod \mathfrak{d}_{N'}$.

For each pair of integers (M, N) with $M \ge N \ge 1$, we define a patching datum $D_{M,N}$ of level N as follows: the statement of the proposition gives a homomorphism $\phi_M : R_\infty \twoheadrightarrow R$ and an S_M-complex D_M. We take

- ϕ to be the composition $R_\infty \overset{\phi_M}{\twoheadrightarrow} R \twoheadrightarrow R/\mathfrak{d}_N$;
- P to be $D_M \otimes_{S_\infty} S_\infty/(\mathfrak{a}_N + \varpi^N) = D_M \otimes_{S_M} S_N/\varpi^N$;
- $\psi : H^{l^0}(P^\square \otimes_{S_\infty^\square} S_\infty^\square/(\mathfrak{a} + \varpi^N)) = H^{l_0}(D_M^\square \otimes_{S_M^\square} \mathcal{O}/\varpi^N) \xrightarrow{\sim} H/\varpi^N$ to be the given isomorphism.

Note that for any ideal I of $\mathcal{O}^\square$ with $(z_1^N, \dots, z_j^N) \subset I \subset (z_1, \dots, z_j)$, we have

$$H^i(P \otimes_{S_\infty} S_{N'} \otimes_{\mathcal{O}} \mathcal{O}^\square/(I + \varpi^{n'})) = H^i(P^\square \otimes_{S_\infty} S_{N'}/\varpi^{n'}) \otimes_{\mathcal{O}^\square} \mathcal{O}^\square/I,$$

and hence this space carries an R_∞-action by assumption (b).

Since there are finitely many patching data of each level $N \ge 1$, up to isomorphism, we can find a sequence of pairs $(M_i, N_i)_{i \ge 1}$ such that

- $M_i \ge N_i$, $M_{i+1} \ge M_i$, and $N_{i+1} \ge N_i$ for all i;
- $D_{M_{i+1},N_{i+1}} \mod \mathfrak{d}_{N_i}$ is isomorphic to D_{M_i,N_i} for all $i \ge 1$.

For each $i \ge 1$, we write $D_{M_i,N_i} = (\phi_i, \psi_i, P_i)$ and we fix an isomorphism between $D_{M_{i+1},N_{i+1}}$ mod $\mathfrak{b}_{N_i}$ and D_{M_i,N_i}. We define

- $\phi_\infty : R_\infty \twoheadrightarrow R$ to be the inverse limit of the ϕ_i;
- $P_\infty^\square := \varprojlim_i P_i^\square/\mathfrak{b}_{N_i}$ where each transition map is the composite of $P_{i+1}^\square/\mathfrak{b}_{N_{i+1}} \twoheadrightarrow P_{i+1}^\square/\mathfrak{b}_{N_i}$ with the isomorphism $P_{i+1}/\mathfrak{b}_{N_i} \xrightarrow{\sim} P_i^\square/\mathfrak{b}_{N_i}$ coming from the chosen isomorphism between $D_{M_{i+1},N_{i+1}} \mod \mathfrak{b}_{N_i}$ and D_{M_i,N_i}.
- ψ_∞ to be the isomorphism of R_∞-modules $H^{l_0}(P_\infty^\square)/\mathfrak{a} = H^{l_0}(P_\infty^\square/\mathfrak{a}) \xrightarrow{\sim} H$ (where R_∞ acts on H via ϕ_∞) arising from the isomorphisms ψ_i.

Then $P_\infty^\square$ is a perfect complex of $S_\infty^\square$-modules concentrated in degrees $0, \dots, l_0$ such that $H^*(P_\infty^\square)$ carries an action of $R_\infty \widehat{\otimes}_{\mathcal{O}} S_\infty^\square$ (extending the action of $S_\infty^\square$). The image of $S_\infty^\square$ in $\mathrm{End}_{\mathcal{O}}(H^*(P_\infty^\square))$ is contained in the image of R_∞. (Use assumption c, and the fact that the image of R_∞ is closed in $\mathrm{End}_{\mathcal{O}}(H^*(P_\infty^\square))$ (with its profinite topology).) It follows that $H^i(P_\infty^\square)$ is a finite R_∞-module for each i. Since $S_\infty^\square$ is formally smooth over $\mathcal{O}$, we can and do choose a homomorphism $\imath : S_\infty^\square \to R_\infty$ in $\mathcal{C}_{\mathcal{O}}$, compatible with the actions of $S_\infty^\square$ and R_∞ on $H^*(P_\infty^\square)$.

Since $\dim_{S_\infty^\square}(H^*(P_\infty^\square)) = \dim_{R_\infty}(H^*(P_\infty^\square))$ and $\dim R_\infty = \dim S_\infty^\square - l_0$, we deduce that $H^*(P_\infty^\square)$ has codimension at *most* l_0 as an $S_\infty^\square$-module. By Lemma 3.2 (with $S = S_\infty^\square$ and $P = P_\infty^\square$) we deduce that $P_\infty^\square$ is a resolution of minimal length of $H^{l_0}(P_\infty^\square)$ and that the latter has $S_\infty^\square$-depth equal to $1 + j + q - l_0$. It follows that

$$\mathrm{depth}_{S_\infty^\square}(H^{l_0}(P_\infty^\square)) = \dim(S_\infty^\square) - l_0 = 1 + j + q - l_0.$$

$\square$

Theorem 3.4. *Keep the notation of the previous theorem and suppose in addition that R_∞ is p-torsion free.*

(1) If R_∞ is formally smooth over $\mathcal{O}$, so $R \simeq \mathcal{O}[[x_1, \dots, x_{q-l_0}]]$, then H is a free R-module.
(2) If $R_\infty[1/p]$ is irreducible, then H is a nearly faithful R-module (in the terminology of [Tay08]*).*

(3) *More generally, H is nearly faithful as an R-module providing that every irreducible component of* $\mathrm{Spec}(R_\infty[1/p])$ *is in the support of* $H^{l_0}(P_\infty)[1/p]$.

Proof. Suppose that $R_\infty \simeq \mathcal{O}[[x_1,\dots,x_{j+q-l_0}]]$. Since $\mathrm{depth}_{R_\infty}(H^{l_0}(P_\infty^\square)) = \dim R_\infty$, applying the Auslander–Buchsbaum formula again, we deduce that $H^{l_0}(P_\infty^\square)$ is free over R_∞. Finally, the existence of the isomorphism $\psi_\infty : H^{l_0}(P_\infty^\square)/\mathfrak{a}H^{l_0}(P_\infty^\square) \xrightarrow{\sim} H$ tells us that $R_\infty/\imath(\mathfrak{a})R_\infty \cong R$ and that R acts freely on H.

For the remaining cases, to show that H is nearly faithful as an R-module, it suffices to show that $H^{l_0}(P_\infty^\square)$ is nearly faithful as an R_∞-module. Since R_∞ is p-torsion free, all its minimal primes have characteristic 0. Thus $H^{l_0}(P_\infty^\square)$ is nearly faithful as an R_∞-module if and only if each irreducible component of $\mathrm{Spec}(R_\infty[1/p])$ lies in the support of $H^{l_0}(P_\infty^\square)[1/p]$. Part (3) follows immediately. For part (2), note that since $\mathrm{depth}_{R_\infty}(H^{l_0}(P_\infty^\square)) = \dim R_\infty$, the support of $H^{l_0}(P_\infty^\square)$ is a union of irreducible components of $\mathrm{Spec}(R_\infty)$ of maximal dimension. Since $H^{l_0}(P_\infty^\square) \neq \{0\}$, the result follows. $\square$

Remark 3.5. *It follows from the proof of the previous theorem that for $H^{l_0}(P_\infty^\square)$ to be nearly faithful as an R_∞-module, it is necessary that R_∞ be equidimensional.*

To implement the level-changing techniques of [Tay08], we will need the following refinement of Theorem 3.3.

Proposition 3.6. *Let S_N and $\mathcal{O}^\square$ be as in Theorem 3.3. Suppose we are given two sets of data $(R_\infty^i, R^i, H^i, T^i, (D_N^i)_{N\geq 1}, (\phi_N^i)_{N\geq 1})$ satisfying assumptions* (1)– (3) *and* (a)– (d) *of Theorem 3.3, for $i = 1, 2$. Suppose also that we are given isomorphisms*

$$\begin{aligned} R_\infty^1/\varpi &\xrightarrow{\sim} R_\infty^2/\varpi \\ R^1/\varpi &\xrightarrow{\sim} R^2/\varpi \\ H^1/\varpi &\xrightarrow{\sim} H^2/\varpi \\ H^{l_0}((D_M^1)^\square \otimes_{S_M} S_N/\varpi) &\xrightarrow{\sim} H^{l_0}((D_M^2)^\square \otimes_{S_M} S_N/\varpi) \end{aligned}$$

for each $M \geq N \geq 0$ and $n \geq 1$, compatible with all actions and such that for each $N \geq 1$ the square

$$\begin{array}{ccc} H^{l_0}(P_N^{1,\square} \otimes_{S_N} \mathcal{O}/\varpi) & \longrightarrow & H^{l_0}(P_N^{2,\square} \otimes_{S_N} \mathcal{O}/\varpi) \\ \downarrow & & \downarrow \\ H^1/\varpi & \longrightarrow & H^2/\varpi \end{array}$$

commutes. Then we can find complexes $P_\infty^{i,\square}$ for $i = 1, 2$ satisfying conclusions (i)– (iv) *of Theorem 3.3 as well as the following additional property:*

- *There is an isomorphism*

$$H^{l_0}(P_\infty^{1,\square})/(\mathfrak{a}+\varpi) \xrightarrow{\sim} H^{l_0}(P_\infty^{2,\square})(\mathfrak{a}+\varpi)$$

  compatible with the actions of R_∞^i/ϖ, $i = 1, 2$, and the isomorphism $R_\infty^1/\varpi \xrightarrow{\sim} R_\infty^2/\varpi$ and such that the square

$$\begin{array}{ccc} H^{l_0}(P_\infty^{1,\square})/(\mathfrak{a}+\varpi) & \longrightarrow & H^{l_0}(P_\infty^{2,\square})/(\mathfrak{a}+\varpi) \\ \downarrow & & \downarrow \\ H^1/\varpi & \longrightarrow & H^2/\varpi \end{array}$$

commutes.

Proof. This can be proved in much the same way as Theorem 3.3; we omit the details. ☐

4. Existence of complexes

In this section, we prove the existence of the appropriate perfect complexes of length l_0 which are required for patching. In both cases, the setting is similar: we have a covering space $X_\Delta(Q) \to X_0(Q)$ of manifolds or an étale map $X_\Delta(Q) \to X_0(Q)$ of schemes over $\mathcal{O}$, each with covering group Δ, which is a finite abelian group of the form $(\mathbf{Z}/p^N\mathbf{Z})^q$. In both cases, the cohomology localized at a maximal ideal $\mathfrak{m}$ of the corresponding Hecke algebra $\mathbf{T}$ is assumed to vanish outside a range of length l_0. The key point is thus to construct complexes of the appropriate length whose size is bounded (in the sense of condition (a) of Theorem 3.3) independently of Q, so that one may apply our patching result.

4.1. **The Betti Case.** A triangulation of $X_0(Q)$ gives rise to a perfect complex $C_\mathcal{O}$ of $\mathcal{O}$-modules. Pulling this triangulation back via the Galois cover $\pi : X_\Delta(Q) \to X_0(Q)$ under the natural projection gives rise to a corresponding perfect complex C of $R = \mathcal{O}[\Delta]$-modules which computes the cohomology of $X_\Delta(Q)$, and such that $C \otimes R/\mathfrak{a} \simeq C_\mathcal{O}$, where $\mathfrak{a}$ is the augmentation ideal of R. Any Hecke operator T gives rise to a map $H^*(C) \to H^*(C)$ of R-modules. Hence (because the complex is perfect) it lifts to a map:

$$T : C \to C$$

of complexes of R-modules.

Lemma 4.1. *Let $C_T := \varinjlim T^n C$. Then C_T is a perfect complex whose cohomology is*

$$\varinjlim T^n H^*(C).$$

Proof. Since R is complete, the functor $M \to \varinjlim T^n M$ is exact, and is in fact a direct summand of M (the other factor being the submodule of M on which T is topologically nilpotent). A direct summand of a projective module is projective, and the equality of homology is likewise an easy computation. ☐

Let $\mathfrak{m}_R$ be the maximal ideal of R.

Lemma 4.2 (Nakayama's Lemma for perfect complexes)**.** *There exists a perfect complex D of free R modules such that*

$$\dim D^n/\mathfrak{m}_R = \dim H^n(C_T \otimes R/\mathfrak{m}_R)$$

for all n. Moreover, the length of D is at most l_0, where l_0 is the range of cohomology groups such that $\varinjlim T^n H^*(C)$ is non-zero.

Proof. By Lemma 1 of Mumford ([Mum08], Ch.II.5), one may find a complex K quasi-isomorphic to C_T such that K is bounded of length l_0, finitely generated, and free (Mumford only guarantees that the final term of the complex is flat, but since R is local, this final term is also free). Assume that the differential d is non-zero modulo $\mathfrak{m}_R$ from degree n to $n+1$. Then by Nakayama's Lemma, there exists a direct sum decomposition

$$K \simeq L \oplus \left(R[n] \xrightarrow{\sim} R[n+1] \right),$$

where L is also a complex of free modules which is clearly quasi-isomorphic to K. Replacing K by L and using induction, we eventually arrive at a complex D so that d is zero modulo $\mathfrak{m}_R$, from which the equality of dimensions follows by Nakayama's Lemma. □

The Hecke algebra $\mathbf{T}$ will be of the form $\mathbf{T}^{\mathrm{an}}[U_x : x \in Q]$ where $\mathbf{T}^{\mathrm{an}}$ is the subalgebra generated by good Hecke operators $T^{(i)}_x$. We say that two maximal ideals of $\mathbf{T}$ give rise to the same Galois representation if they contract to the same ideal of $\mathbf{T}^{\mathrm{an}}$. In order to apply the above lemmas in practice, we define take the Hecke operator T to be:

$$\prod_{Q} \pi_x \circ \prod_{\Omega} (T_x - \eta_x),$$

where Ω is chosen as follows: for any maximal ideal $\mathfrak{n}$ of $\mathbf{T}$ which gives rise to a Galois representation distinct from $\mathfrak{m}$, there exists a good Hecke operator $T^{(i)}_x$ for some prime x and an η_x such that $T^{(i)}_x - \eta_x \in \mathfrak{n}$ but not in $\mathfrak{m}$. Let Ω be a finite set of pairs (T_x, η_x) including one for each $\mathfrak{n}$. For the maximal ideals $\mathfrak{n}$ with the same Galois representation as $\mathfrak{m}$, for all $x|Q$ by construction there will be a projector π_x which commutes with the action of the diamond operators and cuts out the localization at $\mathfrak{m}$. (For example, if $G = \mathrm{GL}(2)$, then π can be taken to be $U_x - \beta_x$ for $x|Q$, where $U_x - \alpha_x \in \mathfrak{m}$ and α_x, β_x are the (distinct) eigenvalues of $\overline{r}(\mathrm{Frob}_x)$). In particular, we have that

$$\lim_{\rightarrow} T^n H^i(\Gamma_\Delta(Q_N), \mathbf{Z}_p) = H^i(\Gamma_\Delta(Q_N), \mathbf{Z}_p)_{\mathfrak{m}}.$$

4.2. **The Coherent Case.** Consider an etale cover $Y := X_\Delta(Q) \to X := X_0(Q)$ with Galois group Δ. Assume, moreover, that X_0 (and thus X_Δ) is flat over $\mathrm{Spec}(\mathcal{O})$. Let $\mathscr{F}$ be a sheaf of $\mathcal{O}_X$-modules which is ϖ-torsion free. Following Nakajima [Nak84], we take an affine covering of X by affine schemes $\{U_\alpha\}$ which are flat over $\mathrm{Spec}(\mathcal{O})$. It follows that the corresponding $\mathcal{O}$-modules $N = \Gamma(U, \mathscr{F})$ are flat, and thus one obtains a Čech complex C of $\mathcal{O}$-flat $\mathcal{O}[\Delta]$-modules (of the form $N \otimes_A B$) computing $H^i(Y, f^*(\mathscr{F}))$.

Lemma 4.3. *Let $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be a finite etale morphism of flat $\mathcal{O}$-algebras. Then, for any ϖ-torsion free A-modules N, $N \otimes_A B$ is a projective $R = \mathcal{O}[\Delta]$-module.*

Proof. Let $M = N \otimes_A B$. Then

$$M/\varpi = (N \otimes_A B)/\varpi = N/\varpi \otimes_{A/\varpi} B/\varpi,$$

and thus M/ϖ is a projective $R/\varpi = k[\Delta]$-module by Lemma 1 of [Nak84]. Since N is ϖ-torsion free, it is $\mathcal{O}$-flat, and thus so is M by the flatness of B. Consider a free resolution $\cdots P_n \to \cdots P_0 \to M \to 0$ of M. Since ϖ is not a zero-divisor in R or M, this sequence remains exact after tensoring with R/ϖ, and hence it follows that

$$\mathrm{Ext}^i_R(M, J) = \mathrm{Ext}^i_{R/\varpi}(M/\varpi, J)$$

for any R/ϖ-module J. Thus $\mathrm{Ext}^i_R(M, J) = 0$ for any R/ϖ-module J since M/ϖ is projective over R/ϖ. In particular, $\mathrm{Ext}^i_R(M, k) = 0$ for all $i > 0$ and thus M is projective. □

It follows that C is a bounded complex of projective R-modules computing $H^i(Y, f^*(\mathscr{F}))$. Applying Lemma 1 of [Mum08], Ch.II.5 again, we may assume in addition that C is finitely generated in each degree (and hence C is perfect). Localizing at some operator T on $H^i(Y, f^*\mathscr{F})$ as above, we obtain a corresponding complex D.

Remark 4.4. A theorem of Kaplansky implies that a projective module P over a local ring R is always free (see [Kap58]).

5. Galois deformations

In this section, we apply our methods to Galois representations of regular weight over number fields. When the relevant local deformation rings are all smooth, the argument is similar to the corresponding result for imaginary quadratic fields in [CG] that corresponds to the case $n = 2$ and $l_0 = 1$. However, in order to prove non-minimal modularity theorems, it is necessary to consider non-smooth rings, following Kisin. The main reference for many of the local computations below is the paper [CHT08].

5.1. **The invariant l_0.** Let F be a number field of signature (r_1, r_2). The invariants l_0 and q_0 may be defined explicitly by the following formula, where "rank" denotes rank over $\mathbf{R}$.

$$\begin{aligned} l_0 &:= r_1\left(\operatorname{rank}(\mathrm{SL}_n(\mathbf{R})) - \operatorname{rank}(\mathrm{SO}_n(\mathbf{R}))\right) + r_2\left(\operatorname{rank}(\mathrm{SL}_n(\mathbf{C})) - \operatorname{rank}(\mathrm{SU}_n(\mathbf{C}))\right) \\ &= \begin{cases} r_1\left(\dfrac{n-1}{2}\right) + r_2(n-1), & n \text{ odd}, \\ r_1\left(\dfrac{n-2}{2}\right) + r_2(n-1), & n \text{ even}. \end{cases} \\ 2q_0 + l_0 &:= r_1\left(\dim(\mathrm{SL}_n(\mathbf{R})) - \dim(\mathrm{SO}_n(\mathbf{R}))\right) + r_2\left(\dim(\mathrm{SL}_n(\mathbf{C})) - \dim(\mathrm{SU}_n(\mathbf{C}))\right) \\ &= r_1\left(n^2 - 1 - \frac{n(n-1)}{2}\right) + r_2\left(2(n^2-1) - (n^2-1)\right). \end{aligned}$$

The invariants l_0 and $2q_0 + l_0$ arise as follows: $2q_0 + l_0$ is the (real) dimension of the locally symmetric space associated to $\mathbb{G} := \mathrm{Res}_{F/\mathbf{Q}}(\mathrm{GL}(n))$, and $[q_0, \ldots, q_0 + l_0]$ is the range such that cuspidal automorphic π for $\mathbb{G}$ which are tempered at ∞ contribute to cuspidal cohomology (see Theorem 6.7, VII, p.226 of [BW80]). (In particular, q_0 is an integer.)

Let V be a representation of G_F of dimension n over a field of characteristic different from 2, and assume that the action of G_{F_v} is *odd* for each $v|\infty$. Explicitly, this is the trivial condition for complex places, and for real places $v|\infty$ says that the action of complex conjugation $c_v \in G_{F_v}$ satisfies $\mathrm{Tr}(c_v) \in \{-1, 0, 1\}$. Then, via an elementary calculation, one has:

$$\sum_{v|\infty} \dim H^0(F_v, \mathrm{ad}^0(V)) = \begin{cases} r_1\left(\dfrac{n^2+1}{2} - 1\right) + r_2(n^2-1), & n \text{ odd}, \\ r_1\left(\dfrac{n^2}{2} - 1\right) + r_2(n^2-1), & n \text{ even}. \end{cases}$$

Thus, in both cases we see that:

$$\sum_{v|\infty} \dim H^0(F_v, \mathrm{ad}^0(V)) = [F : \mathbf{Q}]\frac{n(n-1)}{2} + l_0. \tag{1}$$

5.2. **Deformations of Galois Representations.** Let $p > n$ be a prime that is unramified in F and assume that $\mathrm{Frac}\, W(k)$ contains the image of every embedding $F \hookrightarrow \overline{\mathbf{Q}}_p$. Fix a continuous absolutely irreducible representation:

$$\overline{r} : G_F \to \mathrm{GL}_n(k).$$

We assume that:

- For each $v|p$, $\overline{r}|_{G_v}$ is Fontaine–Laffaille with weights $[0, 1, \ldots, n-1]$ for each $\tau : \mathcal{O}_F \to k$ factoring through $\mathcal{O}_{F_v}$.
- For each $v \nmid p$, $\overline{r}|_{G_v}$ has at worst unipotent ramification and $\mathbf{N}_{F/\mathbf{Q}}(v) \equiv 1 \mod p$ if $\overline{r}$ is ramified at v.
- The restriction $\overline{r}|_{G_v}$ is odd for each $v|\infty$.

We also fix a continuous character

$$\xi : G_F \to \mathcal{O}^\times$$

lifting $\det(\overline{r})$ and with

- $\xi|_{I_v} = \epsilon^{n(n-1)/2}$ for all $v|p$, and
- $\xi|_{I_v} = 1$ for all $v \nmid p$.

Let S_p denote the set of primes of F lying above p. Let R denote a finite set of primes of F disjoint from S_p that contains all primes at which $\overline{r}$ ramifies and is such that $\mathbf{N}_{F/\mathbf{Q}}(v) \equiv 1 \mod p$ for each $v \in R$. Let S_a and Q denote finite sets of primes of F disjoint from each other and from $R \cup S_p$. Finally, let $S = S_p \coprod R \coprod S_a$ and $S_Q = S \coprod Q$. In what follows, R will consist of primes away from p where we allow ramification; Q will consist of Taylor–Wiles primes; and S_a will consist of auxiliary primes which are present to ensure that our level subgroups are sufficiently small.

5.2.1. *Local deformation rings.* For $v|p$, let R_v denote the framed Fontaine–Laffaille $\mathcal{O}$-deformation ring with determinant $\xi|_{G_{F_v}}$ and τ-weights equal to $[0, 1, \ldots, n-1]$ for each $\tau : \mathcal{O}_F \hookrightarrow W(k)$. By [CHT08] Prop. 2.4.3, R_v is formally smooth over $\mathcal{O}$ of relative dimension $n^2 - 1 + [F_v : \mathbf{Q}_p]n(n-1)/2$.

For each $v \in R$, choose a tuple $\chi_v = (\chi_{v,1}, \ldots, \chi_{v,n})$ of distinct characters

$$\chi_{v,i} : I_v \longrightarrow 1 + \mathfrak{m}_{\mathcal{O}} \subset \mathcal{O}^\times$$

such that $\prod_i \chi_{v,i}$ is trivial. We introduce the following framed deformation rings for each $v \in R$:

- Let R_v^1 denote the universal framed $\mathcal{O}$-deformation ring of $\overline{r}|_{G_v}$ corresponding to lifts of determinant ξ and with the property that each element $\sigma \in I_v$ has characteristic polynomial $(X-1)^n$.
- Let $R_v^{\chi_v}$ denote the universal framed $\mathcal{O}$-deformation ring of $\overline{r}|_{G_v}$ corresponding to lifts of determinant ξ and with the property that each element $\sigma \in I_v$ has characteristic polynomial $\prod_i (X - \chi_{v,i}(\sigma))$.

We let

$$\begin{aligned} R_{\mathrm{loc}}^1 &:= \left(\widehat{\bigotimes}_{v \in S_p} R_v\right) \widehat{\bigotimes} \left(\widehat{\bigotimes}_{v \in R} R_v^1\right) \\ R_{\mathrm{loc}}^\chi &:= \left(\widehat{\bigotimes}_{v \in S_p} R_v\right) \widehat{\bigotimes} \left(\widehat{\bigotimes}_{v \in R} R_v^{\chi_v}\right) \end{aligned}$$

Lemma 5.1. *The rings R_{loc}^1 and R_{loc}^χ have the following properties:*

(1) Each of R_{loc}^1 and R_{loc}^χ is p-torsion free and equidimensional of dimension

$$1 + |S_p \cup R|(n^2 - 1) + [F : \mathbf{Q}]\frac{n(n-1)}{2}.$$

(2) We have a natural isomorphism:

$$R_{\mathrm{loc}}^1/\varpi \xrightarrow{\sim} R_{\mathrm{loc}}^\chi/\varpi.$$

(3) The topological space $\mathrm{Spec} R^{\chi}_{\mathrm{loc}}$ *is irreducible.*
(4) Every irreducible component of $\mathrm{Spec} R^{1}_{\mathrm{loc}}/\varpi$ *is contained in a unique irreducible component of* $\mathrm{Spec} R^{1}_{\mathrm{loc}}$.

Proof. This follows from [BLGHT11, Lemma 3.3] using [Tay08, Prop. 3.1] and the properties of the Fontaine–Laffaille rings R_v recalled above. □

For each $v \in Q$, we assume that:

- $\overline{r}|_{G_v} \cong \overline{s}_v \oplus \overline{\psi}_v$ where $\overline{\psi}_v$ is a generalized eigenspace of Frobenius of dimension 1.

Moreover, we let $\mathcal{D}_v$ denote the deformation problem (in the sense of [CHT08, Defn. 2.2.2]) consisting of lifts r of $\overline{r}|_{G_v}$ of determinant ξ and of the form $\rho \cong s_v \oplus \psi_v$ where s_v (resp. ψ_v) lifts $\overline{s}_v$ (resp. $\overline{\psi}_v$) and I_v acts via scalars on s_v and ψ_v. Let

$$L_v \subset H^1(G_v, \mathrm{ad}^0(\overline{r}))$$

denote the Selmer condition determined by all deformations of $\overline{r}|G_v$ to $k[\epsilon]/\epsilon^2$ of type $\mathcal{D}_v$. Then

$$\dim_k L_v - h^0(G_v, \mathrm{ad}^0(\overline{r}) = 1.$$

We assume that $S_a \neq \emptyset$ and that for each $v \in S_a$ we have:

- $H^0(G_v, \mathrm{ad}^0(\overline{r})(1)) = \{0\}$,
- If v lies above the rational prime x, then $[F(\zeta_x) : \mathbf{Q}] > n$.

If

$$L_v := H^1(G_v, \mathrm{ad}^0(\overline{r})),$$

then by local duality and our first assumption on v, we have

$$\dim_k L_v - h^0(G_v, \mathrm{ad}^0(\overline{r}) = 0.$$

5.2.2. *Global deformation rings.* We now consider the following global deformation data:

$$\begin{aligned}\mathcal{S}_Q &= (\overline{r}, \mathcal{O}, S_Q, \xi, (\mathcal{D}_v)_{v \in S_p \cup Q}, (\mathcal{D}^1_v)_{v \in R})\\ \mathcal{S}^{\chi}_Q &= (\overline{r}, \mathcal{O}, S_Q, \xi, (\mathcal{D}_v)_{v \in S_p \cup Q}, (\mathcal{D}^{\chi}_v)_{v \in R}),\end{aligned}$$

where $\mathcal{D}_v$, $\mathcal{D}^1_v$ and $\mathcal{D}^{\chi}_v$ are the local deformation problems determined by the rings R_v, R^1_v and $R^{\chi_v}_v$ for $v \in S_p$ or $v \in R$. A deformation of $\overline{r}$ to an object of $\mathcal{C}_{\mathcal{O}}$ is said to be of type $\mathcal{S}_Q$ (resp. $\mathcal{S}^{\chi}_Q$) if:

(1) it is unramified outside S_Q;
(2) it is of determinant ξ;
(3) for each $v \in S_p \cup Q$, it restricts to a lifting of type $\mathcal{D}_v$ and for $v \in R$, to a lifting of type $\mathcal{D}^1_v$ (resp. $\mathcal{D}^{\chi}_v$).

If $Q = \emptyset$, we will denote $\mathcal{S}_Q$ and $\mathcal{S}^{\chi}_Q$ by $\mathcal{S}$ and $\mathcal{S}^{\chi}$. The functor from $\mathcal{C}_{\mathcal{O}}$ to Sets sending R to the set of deformations of type $\mathcal{S}_Q$ (resp. $\mathcal{S}^{\chi}_Q$) is represented by an object $R_{\mathcal{S}_Q}$ (resp. $R_{\mathcal{S}^{\chi}_Q}$).

We will also need to introduce framing. To this end, let

$$T = S_p \cup R.$$

Let $R^{\square_T}_{\mathcal{S}_Q}$ (resp. $R^{\square_T}_{\mathcal{S}^{\chi}_Q}$) denote the object of $\mathcal{C}_{\mathcal{O}}$ representing the functor sending R in $\mathcal{C}_{\mathcal{O}}$ to the set of deformations of $\overline{r}$ of type $\mathcal{S}_Q$ (resp. $\mathcal{S}^{\chi}_Q$) framed at each $v \in T$. (We refer to Definitions 2.2.1 and 2.2.7 of [CHT08] for the notion of a framed deformation of a given type, replacing the group $\mathcal{G}_n$ of *op. cit.* with GL_n where appropriate.)

We have natural maps

$$\begin{aligned} R_{\mathcal{S}_Q} &\longrightarrow R^{\square_T}_{\mathcal{S}_Q} \\ R^1_{\mathrm{loc}} &\longrightarrow R^{\square_T}_{\mathcal{S}_Q} \end{aligned}$$

coming from the obvious forgetful maps on deformation functors. Similar maps exist for the 'χ-versions' of these rings. The following lemma is immediate:

Lemma 5.2. *The map*

$$R_{\mathcal{S}_Q} \longrightarrow R^{\square_T}_{\mathcal{S}_Q}$$

is formally smooth of relative dimension $n^2|T|-1$. The same statement holds for the corresponding rings of type $\mathcal{S}^\chi_Q$.

We now consider the map $R^1_{\mathrm{loc}} \to R^{\square_T}_{\mathcal{S}_Q}$. For this, we will need to consider the following Selmer groups:

$$\begin{aligned} H^1_{\mathcal{L}(Q),T}(G_F, \mathrm{ad}^0\overline{r}) &:= \ker\left(H^1(G_{F,S_Q}, \mathrm{ad}^0\overline{r}) \to \bigoplus_{x\in T} H^1(G_x, \mathrm{ad}^0\overline{r}) \bigoplus \bigoplus_{x\in Q\cup S_a} H^1(G_x, \mathrm{ad}^0\overline{r})/L_x\right) \\ H^1_{\mathcal{L}(Q)^\perp,T}(G_F, \mathrm{ad}^0\overline{r}(1)) &:= \ker\left(H^1(G_{F,S_Q}, \mathrm{ad}^0\overline{r}(1)) \to \bigoplus_{x\in Q\cup S_a} H^1(G_x, \mathrm{ad}^0\overline{r}(1))/L_x^\perp\right). \end{aligned}$$

Proposition 5.3. *(1) The ring $R^{\square_T}_{\mathcal{S}_Q}$ (resp. $R^{\square_T}_{\mathcal{S}^\chi_Q}$) is a quotient of a power series ring over R^1_{loc} (resp. R^χ_{loc}) in*

$$h^1_{\mathcal{L},T}(G_F, \mathrm{ad}^0\overline{r}) + \sum_{v\in T} h^0(G_v, \mathrm{ad}\overline{r}) - h^0(G_F, \mathrm{ad}\overline{r}) \text{ variables.}$$

(2) We have

$$\begin{aligned} h^1_{\mathcal{L},T}(G_F, \mathrm{ad}^0\overline{r}) &= h^1_{\mathcal{L}^\perp,T}(G_F, \mathrm{ad}^0\overline{r}(1)) + h^0(G_F, \mathrm{ad}^0\overline{r}) - h^0(G_F, \mathrm{ad}^0\overline{r}(1)) \\ &\quad + \sum_{v\in S_a\cup Q} (\dim_k L_v - h^0(G_v, \mathrm{ad}^0\overline{r})) - \sum_{v\in T\cup\{v|\infty\}} h^0(G_v, \mathrm{ad}^0\overline{r}). \end{aligned}$$

Proof. The first part follows from the argument of [Kis09, Lemma 3.2.2] while the second follows from Poitou-Tate duality and the global Euler characteristic formula (c.f. the proof of [Kis09, Prop. 3.2.5]). □

5.3. **The numerical coincidence.** By choosing a set of Taylor–Wiles primes Q to kill the dual Selmer group, one deduces the following.

Proposition 5.4. *Assume that $\overline{r}(G_{F(\zeta_p)})$ is big and let $q \geq h^1_{\mathcal{L}^\perp,T}(G_F, \mathrm{ad}^0\overline{r}(1))$ be an integer. Then for any $N \geq 1$, we can find a tuple $(Q, (\overline{\psi}_v)_{v\in Q})$ where*

(1) Q is a finite set of primes of F disjoint from S with $|Q| = q$.
(2) For each $v \in Q$, we have $\overline{r}|_{G_v} \cong \overline{s}_v \oplus \overline{\psi}_v$ where $\overline{\psi}_v$ is a generalized eigenspace of Frobenius of dimension 1.
(3) For each $v \in Q$, we have $\mathbf{N}_{F/\mathbf{Q}}(v) \equiv 1 \mod p^N$.

(4) *The ring* $R^{\square_T}_{\mathcal{S}_Q}$ *(resp.* $R^{\square_T}_{\mathcal{S}^\chi_Q}$*) is a quotient of a power series ring over* R^1_{loc} *(resp.* R^χ_{loc}*) in*

$$q + |T| - 1 - [F:\mathbf{Q}]\frac{n(n-1)}{2} - l_0.$$

variables.

Proof. Suppose given a tuple $(Q, (\overline{\psi}_v)_{v\in Q})$ satisfying the first three properties. Let $e_v \in \mathrm{ad}\overline{r}$ denote the G_v-equivariant projection onto $\overline{\psi}_v$. Then, as in [CHT08, Prop. 2.5.9] (although we work here with a slightly different deformation problem at each $v \in Q$) we have

$$0 \longrightarrow H^1_{\mathcal{L}(Q)^\perp,T}(G_F, \mathrm{ad}^0\overline{r}(1)) \longrightarrow H^1_{\mathcal{L}^\perp,T}(G_F, \mathrm{ad}^0\overline{r}(1)) \longrightarrow \oplus_{v\in Q} k$$

where the last map is given by $[\phi] \mapsto (\mathrm{tr}(e_v\phi(\mathrm{Frob}_v)))_v$. The argument of [CHT08, Prop. 2.5.9] can then be applied to deduce that one may choose a tuple $(Q, (\overline{\psi}_x)_{x\in Q})$ satisfying the first three required properties and such that

$$H^1_{\mathcal{L}(Q)^\perp,T}(G_F, \mathrm{ad}^0\overline{r}(1)) = \{0\}.$$

The last property then follows from Prop. 5.3, equation (1) and the fact that

$$\dim_k L_v - h^0(G_v, \mathrm{ad}^0\overline{r}) = \begin{cases} 1 & \text{if } v \in Q \\ 0 & \text{if } v \in S_a \end{cases}.$$

$\square$

6. Homology of Arithmetic Quotients

Let $\mathbb{A}$ denote the adeles of $\mathbf{Q}$, and $\mathbb{A}^\infty$ the finite adeles. Similarly, let $\mathbb{A}_F$ and $\mathbb{A}^\infty_F$ denote the adeles and finite adeles of F. Let $\mathbb{G} = \mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}(n)$, and write $G_\infty = \mathbb{G}(\mathbf{R}) = \mathrm{GL}_n(\mathbf{R})^{r_1} \times \mathrm{GL}_n(\mathbf{C})^{r_2}$. Let Z_∞ be the centre of $\mathbb{G}(\mathbf{R})$, and Z^0_∞ the connected component of Z_∞. Furthermore, let K_∞ denote a maximal compact of G_∞ with connected component K^0_∞. For any (sufficiently small) compact open subgroup K of $\mathbb{G}(\mathbb{A}^\infty)$, we may define an arithmetic manifold $Y(K)$ as follows:

$$Y(K) := \mathbb{G}(F)\backslash\mathbb{G}(\mathbb{A})/Z^0_\infty K^0_\infty K.$$

It has dimension $2q_0 + l_0$ in the notation above. We will specifically be interested in the following K. Let $S = S_p \cup R \cup S_a$ and $S_Q = S \cup Q$ be as in Section 5.

6.0.1. *Arithmetic Quotients.* Let $K_Q = \prod_v K_{Q,v}$ and $L_Q = \prod_v L_{Q,v}$ denote the open compact subgroups of $\mathbb{G}(\mathbb{A})$ such that:

(1) If $v \in Q$, $K_{Q,v} = \{g \in \mathrm{GL}_n(\mathcal{O}_v) \mid g \text{ stabilizes } \ell \mod \pi_v\}$ where ℓ is a fixed line in k^n_v.
(2) If $v \in Q$, $L_{Q,v} \subset K_{Q,v}$ is the subgroup of elements which act trivially on ℓ.
(3) If $v \notin S_Q$ $K_{Q,v} = L_{Q,v} = \mathrm{GL}_n(\mathcal{O}_v)$.
(4) If $v \in R$, $K_{Q,v} = L_{Q,v} =$ the Iwahori $\mathrm{Iw}(v)$ subgroup of $\mathrm{GL}_n(\mathcal{O}_v)$ associated to the upper triangular unipotent subgroup.
(5) If $v \in S_a$, $K_{Q,v} = L_{Q,v} = \ker(\mathrm{GL}_n(\mathcal{O}_v) \to \mathrm{GL}_n(k_v))$. (This ensures that K_Q and L_Q are sufficiently small.)

When $Q = \emptyset$, we let $Y = Y(K_\emptyset)$. Otherwise, we define the arithmetic quotients $Y_0(Q)$ and $Y_1(Q)$ to be $Y(L_Q)$ and $Y(K_Q)$ respectively. They are the analogs of the modular curves corresponding to the congruence subgroups consisting of $\Gamma_0(Q)$ and $\Gamma_1(Q)$.

For each $v \in R$, let $\mathrm{Iw}_1(v) \subset \mathrm{Iw}(v)$ denote the pro-v Iwahori. We fix a character

$$\psi_v = \psi_{v,1} \times \cdots \times \psi_{v,n} : \mathrm{Iw}(v)/\mathrm{Iw}_1(v) \cong (k_v^\times)^n \longrightarrow 1 + \mathfrak{m}_{\mathcal{O}} \subset \mathcal{O}^\times.$$

The collection of characters $\psi = (\psi_v)_{v \in R}$ allows us to define a local system of free rank 1 $\mathcal{O}$-modules $\mathcal{O}(\psi)$ on $Y_i(Q)$ for $i = 1, 2$: let $\pi : \widetilde{Y}_i(Q) \to Y_i(Q)$ denote the arithmetic quotient obtained by replacing the subgroup $\mathrm{Iw}(v)$ with $\mathrm{Iw}_1(v)$ for each $v \in R$. Then, a section of $\mathcal{O}(\psi)$ over an open subset $U \subset Y_i(Q)$ is a locally constant function $f : \pi^{-1}(U) \to \mathcal{O}$ such that $f(\gamma u) = \psi(\gamma) f(u)$ for all $\gamma \in \mathrm{Iw}(v)/\mathrm{Iw}_1(v)$. We let $H^i_\psi(Y_i(Q), \mathcal{O})$ and $H_{i,\psi}(Y_i(Q), \mathcal{O})$ denote $H^i(Y_i(Q), \mathcal{O}(\psi))$ and $H_i(Y_i(Q), \mathcal{O}(\psi))$. Note that if $\psi = 1$ is the collection of trivial characters, then $\mathcal{O}(\psi) \cong \mathcal{O}$ and hence $H^i_\psi(Y_i(Q), \mathcal{O}) \cong H^i(Y_i(Q), \mathcal{O})$.

6.0.2. *Hecke Operators.* We recall the construction of the Hecke operators. Let $g \in \mathbb{G}(\mathbb{A}^\infty)$ be an invertible matrix trivial at each place $v \in R$. For $K \subset \mathbb{G}(\mathbb{A}^\infty)$ a compact open subgroup of the form K_Q or L_Q, the Hecke operator $T(g)$ is defined on the homology modules $H_{\bullet,\psi}(Y(K), \mathcal{O})$ by considering the composition:

$$H_{\bullet,\psi}(Y(K), \mathcal{O}) \to H_{\bullet,\psi}(Y(gKg^{-1} \cap K_Q), \mathcal{O}) \to H_{\bullet,\psi}(Y(K \cap g^{-1}Kg), \mathcal{O}) \to H_{\bullet,\psi}(Y(K), \mathcal{O}),$$

the first map coming from the corestriction map, the second coming from the map $Y(gKg^{-1} \cap K, \mathcal{O}) \to Y(K \cap g^{-1}Kg, \mathcal{O})$ induced by right multiplication by g on $\mathbb{G}(\mathbb{A})$ and the third coming from the natural map on homology. The maps on cohomology $H^\bullet_\psi(Y(K), \mathcal{O})$ are defined similarly. (Since, conjecturally, the cohomology of the boundary will vanish after localizing at the relevant $\mathfrak{m}$, we may work either with cohomology or homology, by duality.) The Hecke operators act on $H^\bullet_\psi(Y(K), \mathcal{O})$ but do not preserve the homology of the connected components. Indeed, the action on the component group $A_K := F^\times \backslash \mathbb{A}_F^{\infty,\times} / \det(K)$ of $Y(K)$ is via the determinant map on $\mathbb{G}(\mathbb{A}^\infty)$ and the natural action of $\mathbb{A}_F^{\infty,\times}$ on A_K. For $\alpha \in \mathbb{A}_F^{\infty,\times}$, we define the Hecke operator $T_{\alpha,k}$ by taking

$$g = \mathrm{diag}(\alpha, \alpha, \ldots, 1, \ldots, 1)$$

consisting of k copies of α and $n - k$ copies of 1. We may define the diamond operators $\langle \alpha \rangle$ via the action of the centre $\mathbb{A}_F^{\infty,\times} = \alpha \cdot \mathrm{Id}$ inside $\mathbb{G}(\mathbb{A}^\infty)$. (The operator $\langle \alpha \rangle$ acts non-trivially on the component group via the element $\alpha^n \in \mathbb{A}_F^{\infty,\times}$.) We now define the Hecke algebra.

Definition 6.1. *Let $\mathbf{T}^{\mathrm{an}}_{Q,\psi}$ denote the subring of*

$$\mathrm{End} \bigoplus_{k,n} H^k_\psi(Y_1(Q), \mathcal{O}/\varpi^n)$$

generated by Hecke endomorphisms $T_{\alpha,k}$ and $\langle \alpha \rangle$ for all $k \le n$ and α away from primes in S_Q, and by $\langle \alpha \rangle$ for all α coprime to the level. Let $\mathbf{T}_{Q,\psi}$ denote the $\mathcal{O}$-algebra generated by the same operators together with U_x for x in Q. If $Q = 1$, we write $\mathbf{T}_\psi$ for $\mathbf{T}_{Q,\psi}$.

If $\epsilon \in \mathcal{O}_F^\times$ is a global unit, then T_ϵ acts by the identity. If $\mathfrak{a} \subseteq \mathcal{O}_F$ is an ideal, we may, therefore, define the Hecke operator $T_{\mathfrak{a}}$ as T_α where $\alpha \in \mathbb{A}_F^\times$ represents the ideal $\mathfrak{a}$.

Remark 6.2. It would be more typical to define $\mathbf{T}_{Q,\psi}$ as the subring of endomorphisms of

$$\mathrm{End} \bigoplus_{k} H^k_\psi(Y_1(Q), \mathcal{O}),$$

except that it would not be obvious from this definition that $\mathbf{T}_{Q,\psi}$ acts on $H^k_{\psi}(Y_1(Q), \mathcal{O}/\varpi^n)$ for any n. It may well be true (for the $\mathfrak{m}$ we consider) that $\mathbf{T}_{Q,\psi}$ acts *faithfully* on $H^{q_0+l_0}_{\psi}(Y_1(Q), \mathcal{O})$ — and indeed (at least for $Q = \emptyset$) this (conjecturally) follows when Theorem 1.2 applies and (in addition) R_{loc} is smooth. Whether one can prove this directly is an interesting question. (The claim is obvious when the cohomology occurs in a range of length $l_0 = 0$, and also follows in the case $l_0 = 1$ given known facts about the action of $\mathbf{T}_Q$ on cohomology with K coefficients.)

6.1. **Conjectures on Existence of Galois Representations.** Let $\mathfrak{m}$ denote a maximal ideal of $\mathbf{T}^{\mathrm{an}}_{Q,\psi}$, and let $\mathbf{T}^{\mathrm{an}}_{Q,\psi,\mathfrak{m}}$ denote the completion. It is a local ring which is finite (but not necessarily flat) over $\mathcal{O}$.

Conjecture A. *There exists a semisimple continuous Galois representation* $\overline{r}_{\mathfrak{m}} : G_F \to \mathrm{GL}_n(\mathbf{T}^{\mathrm{an}}_{Q,\psi}/\mathfrak{m})$ *with the following property: if* $\lambda \notin S_Q$ *is a prime of* F*, then* $\overline{r}_{\mathfrak{m}}$ *is unramified at* λ*, and the characteristic polynomial of* $\overline{r}_{\mathfrak{m}}(\mathrm{Frob}_\lambda)$ *is*

$$X^n - T_{\lambda,1}X^{n-1} + \ldots + (-1)^i \mathbf{N}_{F/\mathbf{Q}}(\lambda)^{i(i-1)/2} T_{\lambda,i} X^{n-i} + \ldots + (-1)^n \mathbf{N}_{F/\mathbf{Q}}(\lambda)^{n(n-1)/2} T_{\lambda,n} \in \mathbf{T}^{\mathrm{an}}_{Q,\psi}/\mathfrak{m}[X].$$

(Note that this property determines $\overline{r}_{\mathfrak{m}}$ *uniquely by the Cebotarev density theorem.) If* $\overline{r}_{\mathfrak{m}}$ *is absolutely irreducible, we say that* $\mathfrak{m}$ *is non-Eisenstein. In this case we further predict that there exists a deformation* $r_{\mathfrak{m}} : G_F \to \mathrm{GL}_n(\mathbf{T}^{\mathrm{an}}_{Q,\psi,\mathfrak{m}})$ *of* $\overline{r}_{\mathfrak{m}}$ *unramified outside* S_Q *and such that the characteristic polynomial of* $r_{\mathfrak{m}}(\mathrm{Frob}_\lambda)$ *is given by the same formula as above. In addition, if* $r_{\mathfrak{m}} \cong \overline{r}$ *(where* $\overline{r}$ *is the representation introduced in Section 5.2), we conjecture that* $r_{\mathfrak{m}}$ *enjoys the following properties:*

(1) *If* $v|p$*, then* $r_{\mathfrak{m}}|_{G_v}$ *is Fontaine-Laffaille with all weights equal to* $[0, 1, \ldots, n-1]$.
(2) *If* $v \in Q$*, then* $r_{\mathfrak{m}}|_{G_v}$ *is a lifting of type* $\mathcal{D}_v$ *where* $\mathcal{D}_v$ *is the local deformation problem specified in Section 5.2.1.*
(3) *If* $v \in R$*, then the characteristic polynomial of* $r_{\mathfrak{m}}(\sigma)$ *for each* $\sigma \in I_v$ *is* $(X - \psi_{v,1}(\mathrm{Art}_v^{-1}(\sigma))) \ldots (X - \psi_{v,n}(\mathrm{Art}_v^{-1}(\sigma)))$.
(4) *The localizations* $H^i_{\psi}(Y_1(Q), \mathcal{O}/\varpi^n)_{\mathfrak{m}}$ *vanish unless* $i \in [q_0, \ldots, q_0 + l_0]$.
(5) *For* $v \in Q$*, let* $P_v(X) = (X - \alpha_v)Q_v(X)$ *denote the characteristic polynomial of* $\overline{r}(\mathrm{Frob}_v)$ *where* $\alpha_v = \overline{\psi}_v(\mathrm{Frob}_v)$*. Let* $\mathfrak{m}_Q$ *denote the maximal ideal of* $\mathbf{T}_{Q,\psi}$ *containing* $\mathfrak{m}$ *and* $V_{\varpi_x} - \alpha_x$ *for all* $x|Q$*. Then there is an isomorphism*
$$\prod_{x \in Q} Q_v(V_{\varpi_x}) : H^*_{\psi}(Y, \mathcal{O}/\varpi^n)_{\mathfrak{m}} \xrightarrow{\sim} H^*_{\psi}(Y_0(Q), \mathcal{O}/\varpi^n)_{\mathfrak{m}_Q},$$

It follows that $r_{\mathfrak{m}}$ *is a deformation of* $\overline{r}$ *of type* $\mathcal{S}_Q$ *(resp.* $\mathcal{S}^{\chi}_Q$*) if each* ψ_v *is the trivial character (resp.* $\psi_v = \chi_v$ *for each* $v \in R$*). In this case, we obtain a surjection* $R_{\mathcal{S}_Q} \twoheadrightarrow \mathbf{T}^{\mathrm{an}}_{Q,1,\mathfrak{m}}$ *(resp.* $R_{\mathcal{S}^{\chi}_Q} \twoheadrightarrow \mathbf{T}^{\mathrm{an}}_{Q,\chi,\mathfrak{m}}$*).*

Some form of this conjecture has been suspected to be true at least as far back as the investigations of F. Grunewald in the early 70's (see [Gru72, GHM78]). Related conjectures about the existence of $\overline{r}_{\mathfrak{m}}$ were made for $\mathrm{GL}(n)/\mathbf{Q}$ by Ash [Ash92], and for $\mathrm{GL}(2)/F$ by Figueiredo [Fig99]. One aspect of this conjecture is that it implies that the local properties of the (possibly torsion) Galois representations are captured by the characeristic zero local deformation rings $R^{\square}_v$ for primes v. One might hope that such a conjecture is true in maximal generality, but we feel comfortable making the conjecture in this case because the relevant local deformation rings (including the Fontaine–Laffaille deformation rings) reflect an honest integral theory, which is not necessarily true of all the local deformation rings constructed by

Kisin [Kis09], (although the work of Snowden [Sno] gives hope that at least in the ordinary case that local deformation rings may capture all integral phenomena). By dévissage, conditions 4 and 5 are satisfied if and only if they are satisfied for $n = 1$, e.g., with coefficients in the residue field $k = \mathcal{O}/\varpi$.

The reason for condition 5 of Conjecture A is that the arguments of §3 of [CHT08] (in particular, Lemma 3.2.2 of *ibid.*) often require that the $\mathrm{GL}_n(F_x)$-modules M in question are $\mathcal{O}$-flat. However, it may be possible to remove this condition, we hope to return to this point later (it is also true that slightly weaker hypotheses are sufficient for our arguments). Note, however, that when $n = 2$, this condition 5 is somewhat easier to handle (cf. Lemma 3.4 of [CG]). This is because there are maps:

$$H^*(Y, \mathcal{O}/\varpi^n)^2_{\mathfrak{m}} \to H^*(Y_0(x), \mathcal{O}/\varpi^n)_{\mathfrak{m}} \to H^*(Y, \mathcal{O}/\varpi^n)^2_{\mathfrak{m}}$$

such that the composition is a matrix with invertible determinant (under the assumptions that the eigenvalues of $\overline{r}(\mathrm{Frob}_x)$ are distinct and $\mathbf{N}_{F/\mathbf{Q}}(x) \equiv 1 \mod p$), which induces a splitting of $\mathfrak{m}$-modules.

6.1.1. *Properties of cohomology groups.* Let $\mathfrak{m}$ denote a non-Eisenstein maximal ideal $\mathfrak{m}$ of $\mathbf{T}$. We have natural homomorphisms

$$\mathbf{T}^{\mathrm{an}}_Q \to \mathbf{T}^{\mathrm{an}} = \mathbf{T}_\emptyset, \qquad \mathbf{T}^{\mathrm{an}}_Q \hookrightarrow \mathbf{T}_Q$$

induced by the map $H^\bullet(Y, \mathcal{O}) \to H^\bullet(Y_1(Q), \mathcal{O})$ and by the natural inclusion. The ideal $\mathfrak{m}_\emptyset$ of $\mathbf{T}_\emptyset$ pulls back to an ideal of $\mathbf{T}^{\mathrm{an}}_Q$ which we also denote by $\mathfrak{m}_\emptyset$ in a slight abuse of notation. The ideal $\mathfrak{m}_\emptyset$ may give rise to multiple maximal ideals $\mathfrak{m}$ of $\mathbf{T}_Q$. Suppose that Q contains a finite number of primes x with $N(x) \equiv 1 \mod p$ and such that $\overline{\rho}(\mathrm{Frob}_x)$ has an eigenvalue α_x with multiplicity one. Then there is a unique corresponding ideal $\mathfrak{m}$ of $\mathbf{T}_Q$ containing $U_x - \alpha_x$.

For each N, there is a natural covering map $Y_1(Q_N) \to Y_0(Q_N)$ with Galois group

$$\widetilde{\Delta} := \prod_{x \in Q} (\mathcal{O}_F/x)^\times.$$

Replacing $Y_1(Q_N)$ with $Y_H(Q_N)$ for the quotient $\Delta_N = (\mathbf{Z}/p^N\mathbf{Z})^q$ of $\widetilde{\Delta}$, we obtain a complex D_N of free $S_N = \mathcal{O}[\Delta_N]$-modules of length l_0 computing the complex $H^*(Y_1(Q_N), \mathcal{O})_{\mathfrak{m}}$. Moreover, if (say) one tensors S_N with $\mathcal{O}/\varpi^m$ for some m, then the cohomology of the resulting complex $H^*(Y_1(Q_N), \mathcal{O})_{\mathfrak{m}}$ which has finite length and moreover has an action of $\mathbf{T}$ by definition of $\mathbf{T}$.

6.2. **Modularity Lifting.** In this section we prove our main theorem on modularity lifting.

We assume Conjecture A and the existence of a maximal ideal $\mathfrak{m}$ of $\mathbf{T} := \mathbf{T}_{\emptyset,1}$ with $\overline{r}_{\mathfrak{m}} \cong \overline{r}$. We assume also that $\overline{r}(G(F(\zeta_p)))$ is big.

Since

$$H^*(Y, \mathcal{O}/\varpi) \cong H^*_\chi(Y, \mathcal{O}/\varpi),$$

the ideal $\mathfrak{m}$ induces a maximal ideal of $\mathbf{T}_\chi := \mathbf{T}_{\emptyset,\chi}$, which we also denote by $\mathfrak{m}$ in a slight abuse of notation. By Conjecture A, we have surjections $R_{\mathcal{S}} \twoheadrightarrow \mathbf{T}_{\mathfrak{m}}$ and $R_{\mathcal{S}_\chi} \twoheadrightarrow \mathbf{T}_{\chi,\mathfrak{m}}$.

Theorem 6.3. *If we regard $H^{q_0+l_0}(Y, \mathcal{O})_{\mathfrak{m}}$ as an $R_{\mathcal{S}}$-module via the map $R_{\mathcal{S}} \twoheadrightarrow \mathbf{T}_{\mathfrak{m}}$, then it is a nearly faithful $R_{\mathcal{S}}$-module.*

Proof. We will apply the results of Section 3.1. For each $N \geq 1$, choose a set of Taylor–Wiles primes Q_N satisfying the assumptions of Proposition 5.4 (for some fixed choice of q). Let

$$g = q + |T| - 1 - [F : \mathbf{Q}]\frac{n(n-1)}{2} - l_0$$

be the integer appearing in part (4) of this proposition. We will apply Proposition 3.6 with the following:

- Let $S_\infty = \mathcal{O}[[(\mathbf{Z}_p)^q]]$ and $S_N = \mathcal{O}[\Delta_N]$ as in the statement of Proposition 3.3.
- Let $j = n^2|T| - 1$ and $\mathcal{O}^\square = \mathcal{O}[[z_1, \ldots, z_j]]$.
- Let

  $$\begin{aligned} R^1_\infty &= R^1_{\mathrm{loc}}[[x_1, \ldots, x_g]] \\ R^2_\infty &= R^\chi_{\mathrm{loc}}[[x_1, \ldots, x_g]]. \end{aligned}$$

  Note each R^i_∞ is p-torsion free and equidimensional of dimension $1 + q + j - l_0$ by Lemma 5.1. In addition, we have a natural isomorphism $R^1_\infty/\varpi \xrightarrow{\sim} R^2_\infty/\varpi$.
- Let $(R^1, H^1) = (R_{\mathcal{S}}, H^{q_0+l_0}(Y, \mathcal{O})_\mathfrak{m})$ and $(R^2, H^2) = (R_{\mathcal{S}^\chi}, H^{q_0+l_0}_\chi(Y, \mathcal{O}))$. Note that we have natural compatible isomorphisms $R^1/\varpi \xrightarrow{\sim} R^2/\varpi$ and $H^1/\varpi \xrightarrow{\sim} H^2/\varpi$.
- Let $T = T^1 = T^2 = H^*(Y, \mathcal{O}/\varpi)_\mathfrak{m}$ regarded as a complex with all differentials $d = 0$.
- For each $N \geq 1$, let $Y_{\Delta_N}(Q_N) \to Y_0(Q_N)$ denote the subcover of $Y_1(Q_N) \to Y_0(Q_N)$ with Galois group $\Delta_N = (\mathbf{Z}/p^N)^q$. Let D^1_N (resp. D^2_N) denote a perfect complex of S_N-modules, concentrated in degrees $[q_0, \ldots, q_0 + l_0]$ whose cohomology computes $H^*(Y_{\Delta_N}(Q_N), \mathcal{O})_\mathfrak{m}$ (resp. $H^*_\chi(Y_{\Delta_N}(Q_N), \mathcal{O})_\mathfrak{m}$). We can and do assume that $D^i_N \otimes S_N/\mathfrak{m}_{S_N} \cong T$ for $i = 1, 2$.
- Choose representatives for the universal deformations of type $\mathcal{S}_{Q_N}$ and $\mathcal{S}^\chi_{Q_N}$ which agree modulo ϖ. This gives rise to isomorphisms

  $$\begin{aligned} R^{\square_T}_{\mathcal{S}_{Q_N}} &\xrightarrow{\sim} R_{\mathcal{S}_{Q_N}}[[z_1, \ldots, z_j]] \\ R^{\square_T}_{\mathcal{S}^\chi_{Q_N}} &\xrightarrow{\sim} R_{\mathcal{S}^\chi_{Q_N}}[[z_1, \ldots, z_j]]. \end{aligned}$$

  In the notation of Proposition 3.3, the rings on the right hand side can be written $R^\square_{\mathcal{S}_{Q_N}}$ and $R^\square_{\mathcal{S}^\chi_{Q_N}}$. By Proposition 5.4, we can and do choose surjections $R^1_\infty \twoheadrightarrow R^\square_{\mathcal{S}_{Q_N}}$ and $R^2_\infty \twoheadrightarrow R^\square_{\mathcal{S}^\chi_{Q_N}}$. Composing these with the natural maps $R^\square_{\mathcal{S}_{Q_N}} \twoheadrightarrow R_{\mathcal{S}_{Q_N}} \twoheadrightarrow R_{\mathcal{S}} = R^1$ and $R^\square_{\mathcal{S}^\chi_{Q_N}} \twoheadrightarrow R_{\mathcal{S}^\chi_{Q_N}} \twoheadrightarrow R_{\mathcal{S}^\chi} = R^2$, we obtain surjections $\phi^1_N : R^1_\infty \twoheadrightarrow R^1$ and $\phi^2_N : R^2_\infty \twoheadrightarrow R^2$.

We have now introduced all the necessary input data to Proposition 3.6. We now check that they satisfy the required conditions.

- For each $M \geq N \geq 0$ with $M \geq 1$ and each $n \geq 1$, we have an action of $R_{\mathcal{S}_{Q_N}}$ (resp. $R_{\mathcal{S}^\chi_{Q_N}}$) on the cohomology $H^*(Y_{\Delta_N}(Q_M), \mathcal{O}/\varpi^n)_\mathfrak{m}$ (resp. $H^*_\chi(Y_{\Delta_N}(Q_M), \mathcal{O}/\varpi^n)_\mathfrak{m}$) by Conjecture A. Applying the functor, $X \mapsto X^\square$, and using the surjection $R^1_\infty \twoheadrightarrow R^\square_{\mathcal{S}_{Q_N}}$ (resp. $R^2_\infty \twoheadrightarrow R_{\mathcal{S}^\chi_{Q_N}}$), we obtain an action of R^1_∞ (resp. R^2_∞) on $H^*(D^{1,\square}_M \otimes_{S_M} S_N/\varpi^n)$ (resp. $H^*(D^{2,\square}_M \otimes_{S_M} S_N/\varpi^n)$).

Thus condition (b) of Proposition 3.3 is satisfied for both sets of patching data. Condition (c) follows from Conjecture A, while condition (d) is clear. Finally, we note that we have

isomorphisms

$$H^{l_0}((D^1_M)^{\square}\otimes_{S_M} S_N/\varpi) = H^{l_0}(Y_{\Delta_N}(Q_M),\mathcal{O}/\varpi)^{\square} \xrightarrow{\sim} H^{l_0}_{\chi}(Y_{\Delta_N}(Q_M),\mathcal{O}/\varpi)^{\square} = H^{l_0}((D^2_M)^{\square}\otimes_{S_M} S_N/\varpi)$$

for all $M \ge N \ge 0$ with $M \ge 1$. These isomorphisms are compatible with the actions of R^i_∞ and give rise to the commutative square required by Proposition 3.6.

We have now satisfied all the requirements of Proposition 3.6 and hence we obtain two complexes $P^{1,\square}_\infty$ and $P^{2,\square}_\infty$. By Lemma 5.1, $\mathrm{Spec} R^2_\infty$ is irreducible, and hence by Theorem 3.4 $H^{l_0}(P^{2,\square}_\infty)$ is nearly faithful as an R^2_∞-module. Thus

$$H^{l_0}(P^{1,\square}_\infty)/\varpi \cong H^{l_0}(P^{2,\square}_\infty)/\varpi$$

is nearly faithful over $R^1_\infty/\varpi \xrightarrow{\sim} R^2_\infty/\varpi$. By Lemma 5.1 and [Tay08, Lemma 2.2], it follows that $H^{l_0}(P^{1,\square}_\infty)$ is nearly faithful over R^1_∞ providing that $H^{l_0}(P^{1,\square}_\infty)$ is p-torsion free. However, each associated prime of $H^{l_0}(P^{1,\square}_\infty)$ is a minimal prime of R^1_∞ and by Lemma 5.1, all such primes have characteristic 0. Thus p cannot be a zero divisor on $H^{l_0}(P^{1,\square}_\infty)$ and the result of Taylor applies. By conclusion (iv) of Proposition 3.3 we deduce that $H^1 = H^{q_0+l_0}(Y,\mathcal{O})_{\mathfrak{m}}$ is nearly faithful over $R^1 = R_{\mathcal{S}}$, as required. □

7. Proof of Theorem 1.1

In this section, we prove Theorem 1.1

Proof. Let A be an elliptic curve over a number field K. If A has CM, then the result is well known, so we may assume that $\mathrm{End}_{\mathbf{C}}(A) = \mathbf{Z}$. Let

$$r = \mathrm{Sym}^{2n-1}\rho : G_K \to \mathrm{GL}_{2n}(\mathbf{Q}_p)$$

denote the representation corresponding to the $(2n-1)$th symmetric power of the Tate module of A. To prove Theorem 1.1, it suffices (following, for example, the proof of Theorem 4.2 of [HSBT10]) to prove that for each n, there exists a p such that r is potentially modular. We follow the proof of Theorem 6.4 of [BLGHT11]. (The reason for following the proof of Theorem 6.4 instead of Theorem 6.3 of *ibid.* is that the latter theorem proceeds via compatible families arising from the Dwork family such that $V[\lambda]_t$ is ordinary but not crystalline, which would necessitate a different version of Theorem 1.2.) In particular, we make the following extra hypothesis:

- There exists a prime p which is totally split in K, and such that $p+1$ is divisible by an integer N_2 which is greater than n and prime to the conductor of A. Moreover, the mod-p representation $\overline{\rho}_A : G_K \to \mathrm{GL}_2(\mathbf{F}_p)$ associated to $A[p]$ is surjective, A has good reduction at all $v|p$, and for all primes $v|p$ we have

$$\overline{\rho}_A : G_{\mathbf{Q}_p} \simeq \mathrm{Ind}^{\mathbf{Q}_p}_{\mathbf{Q}_{p^2}} \omega_2.$$

(This is a non-trivial condition on A, we consider the general case below.) It then suffices to find sufficiently large primes p and l, a finite extension L/K, an integer N_2 with $N_2 > n+1$ and $p+1 = N_1N_2$ (as in the statement of Theorem 6.4 of § 4 of [BLGHT11]) and primes λ, λ' of $\mathbf{Q}(\zeta_N)^+$ (with λ dividing p and λ' dividing l) and a point $t \in T_0(L)$ on the Dwork family such that:

1. $V[\lambda]_t \simeq \overline{r}|_{G_L}$,
2. $V[\lambda']_t \simeq \overline{r}'|_{G_L}$, where r' is an ordinary weight 0 representation which induced from G_{LM} for some suitable CM field $M/\mathbf{Q}$ of degree $2n$.
3. p splits completely in L.

(4) A and V are semistable over L.
(5) $\overline{r}|_{G_L}$ and $\overline{r}'|_{G_L}$ satisfy all the hypotheses of Theorem 1.2 with the possible exception of residual modularity.

This can be deduced (as in the proof of Theorem 6.4 of [BLGHT11]) using the theorem of Moret–Bailly (in the form of Prop. 6.2 of *ibid*) and via character building. By construction, the modularity of r follows from two applications of Theorem 1.2, once applied to the λ'-adic representation associated to V (using $\overline{r}'$ and the residual modularity coming from the induction of a Grossencharacter) and once to the λ-adic representation associated to $\mathrm{Sym}^{2n-1}(A)$, using the residual modularity coming from V.

For a general elliptic curve E, we reduce to the previous case as follows. It suffices to find a second elliptic curve A, a number field L/K, and primes p and q such that:

(1) The mod-p representation $\overline{r} = (\mathrm{Sym}^{2n-1}\overline{\rho}_E)|_{G_L}$ satisfies all the hypotheses of Theorem 1.2 with the possible exception of residual modularity.
(2) A and E are semistable over L and have good reduction at all primes dividing p and q.
(3) p and q split completely in L.
(4) $p+1$ is divisible by an integer $N_2 > n+1$ which is prime to the conductor of A.
(5) $E[q] \simeq A[q]$ as G_L-modules, and the corresponding mod-p representation is surjective.
(6) The mod-p representation $\overline{\rho}_A : G_L \to \mathrm{GL}_2(\mathbf{F}_p)$ associated to A is surjective, and $\overline{\rho}_A|_{G_{\mathbf{Q}_p}} \simeq \mathrm{Ind}_{\mathbf{Q}_{p^2}}^{\mathbf{Q}_p}\omega_2$.

This lemma also follows easily from Prop. 6.2 of [BLGHT11], now applied to twists of a modular curve. We deduce as above (using the mod-q representation) that $\mathrm{Sym}^{2n-1}(A)$ is potentially modular over some extension which is unramified at p, and then use Theorem 1.2 once more now at the prime at p to deduce that $\mathrm{Sym}^{2n-1}(E)$ is modular. □

Remark 7.1. It is no doubt possible to also deal with even symmetric powers using the tensor product idea of Harris [Har09].

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